The Importance of the Riemann Zeta Function

Vishal Gupta

1 Introduction and historical background

You never know what premium mathematics is until you have heard of the Riemann zeta function.

– L.B. Seekos

The zeta function was introduced by Euler in the 1700s, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s \in \mathbb{C}$ is a complex number with real part greater than 1, to ensure that the series converges (in fact it converges absolutely).

He discovered the following equation known as the Euler Product form of the Zeta function:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

This suggests a relationship between the zeta function and prime numbers.

Proof: If you imagine expanding out the product and collecting the terms, since

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + \dots)$$

you'll pick up $\frac{1}{n^s}$ exactly once because of the fundamental theorem of arithmetic. The proof essentially follows this intuition but with a finite product as we don't know about its convergence yet. Let $s = \sigma + it$ where $\sigma > 1$. Then

$$\begin{aligned} \left| \zeta(s) - \prod_{p \le x} \frac{1}{1 - p^{-s}} \right| &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \prod_{p \le x} (1 + p^{-s} + p^{-2s} + \dots) \\ &= \left| \sum_{n \in S} n^{-s} \right| \le \sum_{n=x}^{\infty} n^{-\sigma} \to 0 \end{aligned}$$

as $x \to \infty$. Here, S is the set of numbers which involve a prime factor larger than x and in the last inequality, we used the triangle inequality and the fact that $|n^s| = n^\sigma$ as the imaginary part has modulus 1. A corollary of this is that $\zeta(s) \neq 0$.

Proof: We can see that $\prod_{p \leq x} (1 - p^{-s})^{-1} \neq 0$ so

$$\left| \zeta(s) \prod_{p \leq x} \frac{1}{1 - p^{-s}} \right| = \left| \prod_{n > x} \frac{1}{1 - p^{-s}} \right| \geq 1 - \left| \sum_{n \in S} \frac{1}{n^s} \right| \to 1$$

as $x \to \infty$.

Knowledge of the divergence of the harmonic series tells you that $\zeta(s)$ diverges as $s \to 1$. Through the Euler product, we can then relate the harmonic series to the corresponding product over primes - in fact we can use $H_n \sim \log n$ and take the log of the product (a standard way to compare infinite products to infinite series) and find the non trivial result that

$$\sum_{p \leq x} \frac{1}{p} \sim \log \log x$$

which also shows that the sum diverges, albeit very slowly. This is a strengthening of Euclid's classical result that there are infinitely many primes.

Proof: Taking logarithms and using the Taylor series of $\log(1 - x)$, we have

$$\begin{aligned}
\log \left(\prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} \right) &= - \sum_{p \leq x} \log \left(1 - \frac{1}{p} \right) \\
&= \sum_{p \leq x} \sum_{k=1}^{\infty} \frac{1}{kp^k} \\
&= \sum_{p \leq x} \frac{1}{p} + \sum_{p \leq x} \sum_{k=2}^{\infty} \frac{1}{kp^k}
\end{aligned}$$

Analysing the second term,

$$\begin{aligned}
\sum_{p \leq x} \sum_{k=2}^{\infty} \frac{1}{kp^k} &\leq \sum_{p \leq x} \sum_{k=2}^{\infty} \frac{1}{p^k} \\
&= \sum_{p \leq x} \frac{p^{-2}}{1 - p^{-1}} = \sum_{p \leq x} \frac{1}{p(p-1)} \leq \sum_{n=1}^{\infty} \frac{1}{n(n-1)} = 1 < \infty
\end{aligned}$$

Now we know that the product behaves like $\log x$ for large x because of the harmonic series so we have the result.

Another interesting subset is the twin primes - primes p such that $p+2$ is a prime. It turns out the sum over twin primes converges to a number known as Brun's constant:

$$\sum_{p,p+2 \text{ prime}} \frac{1}{p} + \frac{1}{p+2} \approx 1.90216...$$

This was proven by Brun in 1919 and also has historical importance as the introduction of sieve methods, an important tool in analytic number theory. The convergence of this sum means we don't learn about whether there are infinitely many twin primes, which is still a famous open problem. But there has been recent progress on this - particularly the work by Yitang Zhang and James Maynard in 2013.

2 Enter Riemann

Given the previous results, studying the ζ function will likely tell us more about primes. When studying a complex function, analytic continuation often is useful to gain new information about the original function - it often paints the function in a new light, making some properties easier to see. An attempt to analytically continue the Riemann zeta function can be made by expressing the sum as an integral of a fractional part which gives, for $\mathrm{Re}(s) > 1$

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\mathrm{d}x = -s\int_0^\infty \frac{\{x\}}{x^{s+1}}\mathrm{d}x$$

These integrals converge for $0 < \mathrm{Re}(s) < 1$ and it agrees with our original definition for $\mathrm{Re}(s) > 1$ so we can take this to extend the definition of the Riemann Zeta function. This formula also shows us that ζ's only pole is a simple pole at $s = 1$. This continuation doesn't extend to all of $\mathbb{C}$ though.

In 1859, Riemann published his only paper on number theory which had a very significant impact on mathematics. Riemann extended the definition of the zeta function to the rest of the complex plane by establishing that the functional equation

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)$$

Proof: The integral representation makes for a fairly simple derivation of the functional equation. Using the Fourier series of $\{x\}$, we have

$$\begin{aligned}
(2^s - 1)\frac{\zeta(s)}{s} &= \int_0^\infty \frac{\{x\} - \{2x\}}{x^{s+1}} \mathrm{d}x \\
&= \int_0^\infty x^{-s-1} \sum_{n=1}^\infty \frac{\sin(4\pi nx) - \sin(2\pi nx)}{n\pi} \mathrm{d}x \\
&= \sum_{n=1}^\infty \frac{1}{n\pi} \int_0^\infty x^{-s-1}(\sin(4\pi nx) - \sin(2\pi nx))\mathrm{d}x \\
&= \sum_{n=1}^\infty \frac{1}{n\pi}(2^s - 2^{2s})\pi^{s-1}\Gamma(-s)\sin\frac{\pi s}{2}\zeta(1-s)
\end{aligned}$$

where the exchange of the sum and integral is justified because the partial sums are uniformly bounded and x^{-s-1} is integrable on that range. Cancelling down, we have the functional equation.

This extends the zeta function to a meromorphic function to the rest of the complex plane, where $\Gamma(s)$ is the Gamma function, for $s \in \mathbb{C}$. This then shows us that $\zeta(-2n) = 0$ for $n \in \mathbb{Z}_{\geq 0}$ because $\sin(-\pi n) = 0$ - these are called the trivial zeroes.

We know that for $\mathrm{Re}(s) > 1$, $\zeta(s) \neq 0$ so, by the functional equation that the zeroes of the zeta function are restricted to the strip $0 \leq \mathrm{Re}(s) \leq 1$ - this is known as the critical strip. Also if $s \in \mathbb{C}$ isn't a negative even integer, if $\zeta(s) = 0$, $\zeta(1-s) = \zeta(\overline{s}) = 0$ - the first is by the functional equation and the second because $\zeta(s) = \overline{\zeta(s)}$ because they agree on $\mathrm{Re}(s) > 1$ so they are equal by the principle of isolated zeroes. The Riemann Hypothesis suggests only one of these symmetries is non trivial - that all of the zeroes are on the line $\mathrm{Re}(s) = \frac{1}{2}$. This is known as the most important problem in mathematics and has many implications for number theory - why?

3 What ζ tells us about Number Theory

We define some the prime counting function, the Chebyshev function and the von Mangoldt function:

$$\pi(x) = \sum_{p \leq x:\ p \text{ prime}} 1 \quad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ and} \\ 0 & \text{otherwise} \end{cases} \quad \psi(x) = \sum_{n \leq x} \Lambda(n)$$

While $\pi(x)$ is the intuitive way to count primes, counting primes with a log weight and including prime powers turns out to be nicer. We can use partial summation to move between the two. As a demonstration of this, the Chebyshev function can be expressed as a sum over the non trivial zeroes ρ of the Riemann zeta function:

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)$$

This formula, known as the Riemann-von Mangoldt explicit formula, is remarkable and the proof is fairly involved - it can be found in [1] on page 31. A version with an error term is the following where x isn't an integer and $T \geq 1$ is some parameter,

$$\psi(x) = x - \sum_{\rho:\, |\gamma| \leq T} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) + O\left(\frac{x}{T}\left(\log(xT)^2 + \frac{\log x}{\langle x \rangle}\right)\right)$$

where $\rho = \beta + i\gamma$ and $\langle x \rangle$ is the distance from x to the nearest prime power.

With knowledge about the properties of the zeroes, we can deduce information about $\psi(x)$ and so, prime numbers. An important question is 'how many prime numbers are there?' which we quantify with $\pi(x)$ - in particular, the growth rate of $\pi(x)$. The prime number theorem proposes the answer to this question, namely that

$$\pi(x) \sim \frac{x}{\log x}$$

The prime number theorem is equivalent to the statement that $\psi(x) \sim x$ with partial summation. Using the explicit formula, we see that this is equivalent to the claim that

$$\lim_{x\to\infty} \frac{-\sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)}{x} = -\lim_{x\to\infty} \sum_\rho \frac{x^{\rho-1}}{\rho} = 0$$

Now if the limit could be taken term wise, it suffices to prove that $x^{\rho-1} \to 0$ which happens if $\mathrm{Re}(\rho) < 1$, then we get the result. We already know that $\mathrm{Re}(\rho) \leq 1$ so the Prime Number Theorem follows from being able to take this limit termwise and proving that $\mathrm{Re}(\rho) \neq 1$. This is the approach taken by Hadamard and de la Vallée Poussin independently in 1896.

Assuming the Riemann Hypothesis - that $\mathrm{Re}(\rho) = \frac{1}{2}$, we can improve this result by providing the error term of this approximation.

$$\psi(x) = x + O(\sqrt{x}(\log x)^2)$$

Proof: We follow the proof in [1] on page 32. Assume that $x \geq 2$ isn't an integer and that $\langle x \rangle \geq 1$ - these assumptions change the outcome by a term of at most $O(\log x)$ which can be included in the error term. Let $T \geq 1$ be some parameter which will be chosen later. Then the explicit formula with error term gives

$$\psi(x) = x + O\left(1 + \sum_{\rho:\, |\gamma| \leq T} \frac{x^\rho}{\rho} + \frac{x}{T}\log(xT)^2\right)$$

Assuming the Riemann Hypothesis, we get

$$\sum_{\rho:\, |\gamma| \leq T} \frac{x^\rho}{\rho} \leq \sqrt{x} \sum_{\rho:\, |\gamma| \leq T} \frac{1}{|\rho|}$$

From the next section we'll learn that the number of zeroes in the region $n \leq t \leq n+1$ is $O(\log n)$ and in that region, we have $|\rho| > n$ so that

$$\sum_{\rho:\,|\gamma|\leq T} \frac{1}{|\rho|} \leq 1 + \sum_{2\leq n\leq T+1} \frac{\log n}{n} \leq (\log T)^2$$

Therfore the error term is bounded by $1 + \sqrt{x}(\log T)^2 + \dfrac{x}{T}\log(xT)^2$. Now we can choose $T = x$ and the result follows.

In fact, the Riemann Hypothesis is equivalent to the statement that

$$\psi(x) = x + O(x^{\frac{1}{2}+\varepsilon})$$

for all $\epsilon > 0$. $\psi(x)$ is determined by the distribution of the primes - our interest in the ζ function was that it tells us about the distribution of the primes but this shows us ζ is similarly determined by knowledge of the distribution of the primes - they are inextricably linked.

4 The number of zeros of ζ

How many zeroes does ζ have? How do we know it has any zeroes? We used this result in the proof of our bound dependent on the Riemann Hypothesis (maybe just name it). Let $N(T)$ be the number of zeroes $\rho = \beta + i\gamma$ in the region $0 \leq \beta \leq 1, 0 \leq \gamma \leq T$. Then we have the following:

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

Proof: We follow the proof in [1] on Page 38, using the argument principle on the region specified above. From before, ζ has a pole at $s = 1$. To avoid a pole on the region of integration, we introduce the Riemann ξ function which has the same zeroes as ζ but is entire

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

The functional equation for ξ is $\xi(s) = \xi(1-s)$ and ξ has no poles and any zero of ξ is a zero of ζ. Without loss of generality, assume that there isn't a zero at height T. Then by the argument principle,

$$N(T) = \frac{1}{2\pi i}\int_C \frac{\xi'(s)}{\xi(s)}\mathrm{d}s$$

where the contour C is a rectangle between -1, 2, 2+iT, -1+iT. Split the contour by a vertical line at $\sigma = \frac{1}{2}$. Then by the functional equation, $\dfrac{\xi'(s)}{\xi(s)} = -\dfrac{\xi'(1-s)}{\xi(1-s)}$

so the integral on the left hand piece is equal to the integral on the contour from $\frac{1}{2} - iT$ to $2 - iT$ to 2 to $\frac{1}{2}$. Integrals cancel, leaving us with

$$N(T) = \int_{C'} \frac{\xi'(s)}{\xi(s)} \mathrm{d}s$$

where C' consists of three line segments from $\frac{1}{2} - iT$ to $2 - iT$ to $2 + iT$ to $\frac{1}{2} + iT$. We also have

$$\frac{\xi'(s)}{\xi(s)} = \frac{1}{s} + \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2}\frac{\Gamma'\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} - \frac{s}{2}\log\pi$$

As long as we don't cross any branch cuts containing zeroes of $f(s)$, the derivative of $\log f$ is $\frac{f'}{f}$ so we can use the fundamental theorem of calculus. This yields

$$N(T) = \frac{1}{2\pi i}\left(\log s + \log(s-1) + \log\zeta(s) + \log\Gamma\left(\frac{s}{2}\right) - \frac{s}{2}\log\pi\right)_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}$$

The first term contributes $\frac{1}{2} + O(T)$, the final term contributes $-\frac{\log\pi}{2\pi}T$.

For Γ, using Stirling's formula we get

$$\log\Gamma(s) = s\log s - s + O(\log s)$$

at $s = \frac{1}{4} \pm \frac{iT}{2}$. Since the argument of $\frac{1}{4} + \frac{iT}{2}$ is $\frac{\pi}{2} + O(\frac{1}{T})$, we get

$$\begin{aligned}\frac{1}{2\pi i}\left(\log\Gamma\left(\frac{s}{2}\right)\right)_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} &= \frac{1}{2\pi i}\left(iT\log\left|\frac{1}{4} + \frac{iT}{2}\right| - iT + O(\log T)\right) \\ &= \frac{T}{4\pi}\log\left(\frac{1}{16} + \frac{T^2}{4}\right) - \frac{T}{2\pi} + O(\log t) \\ &= \frac{T}{2\pi}\log\left(\frac{T}{2}\right) - \frac{T}{2\pi} + O(\log T)\end{aligned}$$

Finally, we have to analyse the $\log\zeta$ term. The real parts cancel so we only need to consider $\operatorname{Im}\log\zeta(\frac{1}{2} \pm iT) = O(\log T)$. We have that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho\colon |\gamma - T| \leq 1} \frac{1}{s-\rho} + O(\log T)$$

There isn't any zero at height T so by inegrating, we get

$$\operatorname{Im}\left(\int_{\frac{1}{2}}^{2} \frac{\zeta'(\sigma + iT)}{\zeta(\sigma + iT)}\right)\mathrm{d}\sigma = \operatorname{Im}(\log\zeta(2+iT) - \log\zeta(\tfrac{1}{2} + iT))$$

The first term is $O(1)$, using the formula $\log\zeta(s) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}n^{-s}$. The left hand side is

$$-\sum_{\rho\ |\gamma-T|\leq 1}\int_{\frac{1}{2}}^{2}\mathrm{Im}\left(\frac{1}{\sigma+iT-\rho}\right)\mathrm{d}\sigma + O(\log T)$$

To bound each summand for $\rho = \beta + i\gamma$, write it as

$$\int_{\frac{1}{2}}^{2}\frac{\gamma - T}{(\sigma-\beta)^2+(\gamma-T)^2}\mathrm{d}\sigma$$

Changing variables via $\sigma - \beta = u(\gamma - T)$ (we have $\gamma - T \neq 0$), this integral is at most

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{1+u^2} = \pi = O(1)$$

So there are $O(\log T)$ summands, each of size $O(1)$ so we have $\mathrm{Im}\log\zeta(\frac{1}{2}+iT) = O(\log T)$. Putting everything together, we get the final result,

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

This formula tells us that there are zeroes - in fact infinitely many of them! It was proved by Hardy in 1914 that there are infinitely many zeroes on the critical line.

5 Conclusion

The Riemann Zeta function has deep connections with properties of primes - understanding prime numbers is a major goal for number theory so the Riemann Zeta function has a very important role in number theory. It's reasonable to expect that any function with similar properties could also have number theoretic value. This motivates the definition of an L-function, a function of the form $L(s) = \sum_{n\geq 1}a_n n^{-s}$ (a Dirichlet series) which has similar properties, an Euler product a functional equation and (conjecturally) a meromorphic continuation, like the ζ function. These L-functions are an important object of study in number theory and have connections to other areas of mathematics. Another example of an important class of L-functions are Dirichlet L-functions. Just like the Riemann Zeta function, it has an Euler product, a functional equation, a meromorphic continuation to the complex plane and Dirichlet showed that they are non-zero when $s = 1$. This was used to show that there are infinitely many primes in arithmetic progressions of the form $a + nd$ where a and d are coprime - another deep number theoretic result. I hope this article illuminated some of the properties of the Riemann Zeta function and its importance in number theory, a seemingly disconnected field from complex analysis.

6 Further Reading

This article was heavily inspired by Dr Bloom's Part III Analytic Number Theory course in 2020 with several proofs being lifted from those notes. Edwards' text 'Riemann's Zeta Function' goes into much further detail on the zeta function such as proving Hardy's theorem that there are infinitely many zeroes on the critical line and improvements of this result. For general reading in analytic number theory, Davenport's 'Multiplicative Number Theory' covers many of the main topics.

7 Bibliography

1. Bloom, T. (2020) Part III Analytic Number Theory, Available here.
2. Edwards, H.M. (1974) Riemann's Zeta Function. New York, Academic Press.