Maths Interview Questions

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These questions are from my website. I have some more stuff you might find useful there.

Questions

1. Suppose there's an invisible rabbit which is moving along a train track from an unknown initial position, and it jumps by an unknown amount every second. Every second, before it jumps, you can pick a point on the train track to try and catch the rabbit. Prove that you can catch it in a finite time. (This question is way harder than the others, so probably try every other question first.)
2. Suppose we have a 6-sided die and roll it 6 times. What is the probability that the rolls are strictly increasing?
3. Consider a polynomial with integer coefficients. Is it always prime for every positive integer input?
4. Evaluate the limit of $\frac{n+n^2+\cdots+n^n}{1^n+2^n+\cdots+n^n}$ as n goes to infinity.
5. Sketch $x^y = y^x$.
6. Sketch $y = \ln(\sin x)$ for $0 < x < \pi$ and $y = \ln(\cos x)$ for $0 < x < \frac{\pi}{2}$. Evaluate the integral of $\ln(\sin x)$ from 0 to $\frac{\pi}{2}$.
7. Take 5 points in an equilateral triangle of side length 1. Prove that there are two of them at a distance of no greater than 0.5.
8. Consider a square of side length 2. If we have 9 points inside it, do three of the points necessarily form a triangle of area less than $\frac{1}{2}$?
9. A grid of size $2 \times n$ is covered with 2×1 dominoes. How many ways are there to do this? What about a grid of size $3 \times n$ and 3×1 dominoes?
10. Given 5 points on a sphere, can we split the sphere in two so that 4 points lie on one of the hemispheres?
11. Solve the equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ for positive integers x, y, z.
12. We roll a 6-sided die n times. What's the probability that all faces have appeared?
13. There are n ants on a rope of length 10m. They all move at a speed of 10 m/s. If two ants bump into each other on the rope, they instantly reverse their direction and move at the same speed. The ants can be facing in any direction. If an ant gets to the end of the rope, then it falls off. Prove that they all fall off eventually. Can you give an upper bound on the amount of time until all the ants fall off the rope?
14. There are 50 ants on a 10m line. The 25 left-most ants are moving right, and the 25 right-most ants are moving left. When 2 ants collide, they will both reverse direction. How many collisions will there have been in total once all ants have fallen off the end of the line?

15. You have n coins $C_1, C_2, \ldots, C_n$. Each coin C_k is biased so that tossing a heads with coin C_k has a probability of $\frac{1}{2k+1}$. If all n coins are tossed, what is the probability that the number of heads is odd?
16. Is there an infinite sequence of non-zero real numbers $a_1, a_2, \ldots$ such that $a_1^m + a_2^m + \ldots = m$ for every positive integer m?
17. Find a positive integer a such that $n^4 + a$ isn't prime for all integers n.
18. Seven different real numbers are chosen. Prove that there are two of them, a and b, such that $\frac{a-b}{1+ab} < \frac{1}{\sqrt{3}}$.
19. Let $n > 6$ be an integer such that $n-1$ and $n+1$ are prime. Prove that $720 \mid n^2(n^2+16)$.
20. What is the expected number of rolls of a 6-sided die for all faces to appear at least once? What about an n-sided die?
21. You play a game with a biased coin. If you get a head, you get a pound and the game continues; if you get a tails, the game ends (you keep your money from any past heads). What is the expected value? Can you do it with conditional expectation?

Hints

Q1

Hint 1: Introduce variables x and y for the initial starting position and the amount the rabbit jumps per second respectively. Why does knowing x and y suffice?

Hint 2: Plot on a graph.

Q2

Hint 1: Think about the number of sets for which this can happen.

Q3

Hint 1: All cases are instant except if you have a + or -1 as the constant term. Can you change your polynomial with a problematic constant term to one which doesn't have this problem?

Hint 2: Why can we always use a translation and obtain a constant term which isn't 1 or -1?

Q4

Hint 1: Dividing top and bottom by n^n will help.

Hint 2: Recall the limit definition of e^x (you can try to prove it if you like).

Q5

Hint 1: Upon rearranging you'll find that $\frac{y}{\ln y} = \frac{x}{\ln x}$.

Hint 2: Sketch the graph of $\frac{x}{\ln x}$ and you'll see the number of solutions to the equation $\frac{y}{\ln y} = \frac{x}{\ln x}$ in the different cases. This will let you draw a sketch—it doesn't need to be exactly correct, remember, so you don't need to work out the equations, just sketch it.

Q6

Hint 1: Add together the analogous integral for $\ln(\cos x)$.

Hint 2: Use the double angle formula.

Q7

Hint 1: If we have two points within one triangle, what's the maximum distance between them?

Hint 2: If we split this up into 4 congruent shapes what happens?

Q8

Hint 1: What's the maximum area of a triangle within a square of a given side length x?

Hint 2: Can you find a smaller area in the square so that there's always three points within it?

Hint 3: Split up the square.

Q9

Hint 1: How many ways to do it for $n = 2, 3, 4$? Trying small numbers can help you spot the pattern.

Hint 2: Think about the different situations you can get after placing some dominoes down—does it reduce to a smaller case?

Hint 3: Do you recognize the sequence made from the recurrence relation? Work out the initial conditions.

Hint 4: For the next ones, $3 \times n$ with 3×1 and 2×1 dominoes, consider what possible cases arise from how you can place the dominoes and which smaller cases that reduces to to make another recursive formula.

Q10

Hint 1: 5 points and 2 hemispheres and proving something for all configurations of points—this is set up like a pigeonhole principle problem!

Hint 2: Notice that it says “on” the hemisphere, not “in”!

Q11

Hint 1: We’re dealing with integers only, so it’s useful to get rid of the fractions so we get an equation with two integers equal to each other.

Hint 2: Without loss of generality, bound them.

Q12

Hint 1: Try to count the number of times you see all faces out of n rolls and divide it by the number of possibilities.

Hint 2: Try using the principle of inclusion-exclusion to count it.

Hint 3: Try using inclusion-exclusion to count the number of possibilities where you don’t get all faces.

Q13

Hint 1: Consider the ant at the end.

(Another method)

Hint 2: Aren’t the ants indistinguishable?

Q14

Hint 1: Try not to think about particular cases/individual ants—try looking at the problem as a whole.

Hint 2: Aren’t the ants indistinguishable?

Hint 3: So we can think of the ants as just passing through each other.

Q15

Hint 1: Imagine the last coin.

Hint 2: Define P_n to be the desired probability. Can you use a recursion?

Q16

Hint 1: Assume it is true. Can you use a bounding argument?

Hint 2: What if they were all less than 1?

Hint 3: What if one of them was greater than 1?

Q17

Hint 1: For something to not be prime, it means there are two numbers multiplying together to make the number you have. For this, it might be useful if we can find a factorization.

Hint 2: Solve the equation $n^4 = -a$ fully.

Hint 3: Can you pair up the complex factorization into giving real terms?

Hint 4: Make sure that your factorization can never give a 1 or -1 so that it isn't prime.

Q18

Hint 1: What function does that expression $\frac{a-b}{1+ab}$ remind you of? And the $\sqrt{3}$?

Hint 2: Seven numbers—looks like a pigeonhole problem

Hint 3: We can write our numbers as $\tan x$ for x in some interval.

Q19

Hint 1: Prime factorise 720 and consider divisibility by each prime factor.

Q20

Hint 1: Think of each step individually.

Hint 2: Use the linearity of expectation.

Hint 3: If i faces have already appeared, then you need $n - i$ to appear. Can use a geometric distribution. An extra for this problem would be to derive the geometric distribution and calculate its expected value.

Q21

Hint 1: Probability of n pounds is the probability of n CONSECUTIVE heads, so work out that and add on the extra required tails by and rule. Call the probabilities h and t for heads and tails respectively.

Hint 2: The thing is an infinite series! Is it similar to any others you know?

Hint 3: Either write out or differentiate.

(For conditional expectation)

Hint 4: $E(X) = \sum A_i E(X|A_i) \cdot P(A_i)$ for countably many events.