Integrals without Integrating

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Introduction

A definite integral is defined to be the area under a curve in A level Maths. Then you state the fundamental theorem of calculus, namely

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\mathrm{d}x\right) = f(x) \tag{1}$$

i.e 'integration is the opposite of differentiation'. From that point, the area definition is used as the endpoint of the calculation - you calculate the integral and now you know the area under the curve. How about we turn this on its head? We can think about the area under the curve to calculate integrals instead. This turns out to be quite powerful, it lets us calculate integrals we couldn't previously do (or at least do them much more easily!) and mirrors the historical development of the integral calculus.

Graphs

The first tool at our disposal are graphs. Some integrals can be calculated quite easily with just graphs - for example

$$\int_0^a 1\mathrm{d}x = a, \int_0^a x\mathrm{d}x = \frac{a^2}{2} \tag{2}$$

can be seen by drawing the graph and finding the area of the resulting shape - in this case a rectangle and a triangle. We can sometimes even do integrals with no calculation at all; from a graph you can see that

$$\int_{-a}^a x\mathrm{d}x = 0 \tag{3}$$

for any number a because the two triangles formed cancel - the one below the axis has negative area and the other has positive area.

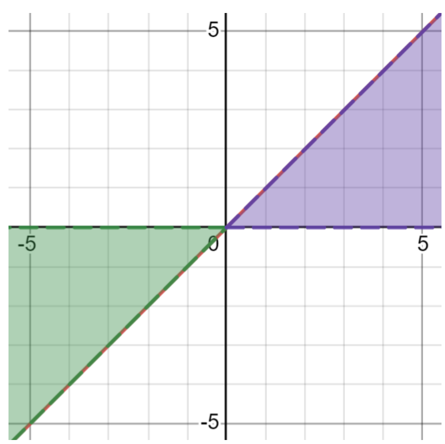

Figure 1: The two regions which cancel

We can see this algebraically too; if $f(x) = x$ then $f(-x) = -x = -f(x)$ i.e the function takes the same values but the negative one - you can think of it as each point cancels its negative out. With this idea we could have found (3) without any graph at all. Functions which satisfy $f(x) = f(-x)$ are called odd functions and you've encountered a lot of them throughout your study, for example $x^3, \sin(x), \tan(x)$. Now we can say

$$\int_{-a}^a x^3\mathrm{d}x = 0 \quad \int_{-a}^a \sin(x) = 0 \quad \int_{-1}^1 \tan(x)\mathrm{d}x$$

A question you can think about is, why can't we do any a for $\tan(x)$ - why did I choose $a = 1$?

We can do more integrals by thinking about the shapes. An example is the integral

$$\int_0^1 \sqrt{1-x^2}\mathrm{d}x$$

Thinking about this, if we have $y = \sqrt{1-x^2}$ then we have $y^2 = 1 - x^2 \implies x^2 + y^2 = 1$ - the equation of a circle. Of course though, we have $0 < x < 1$ as dictated by the bounds of our integral leaving us with $0 < y < 1$. This means the graph is just the circle in the positive quarter of the plane, which is exactly what we see when we sketch the graph:

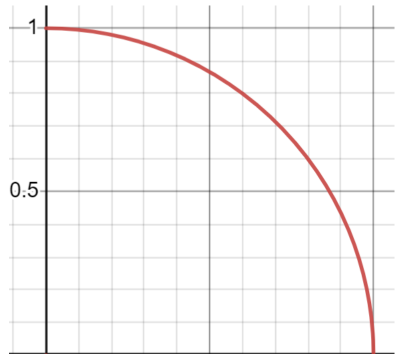

Figure 2: The graph of $y = \sqrt{1-x^2}$ for $0 < x < 1$

We can then see instantly that

$$\int_0^1 \sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{4}$$

Normally you need something called integration by substitution to do this; something which appears in A level maths.

These ideas come together in the following meme:

Figure 3: The integral you're faced with when trying to access wifi at Nanjing University of Aeronautics and Astronautics

It looks pretty nasty but with these ideas it's very simple. First the boundaries look exactly like what we've done with the odd functions so it's good to start there. Doing that, you can notice that $x^3 \cos\left(\frac{x}{2}\right)\sqrt{4-x^2}$ is an odd function - x^3 is odd while $\cos\left(\frac{x}{2}\right)$ and $\sqrt{4-x^2}$ is even. So the scary part of the integral disappears, leaving us with just

$$\frac{1}{2}\int_{-2}^{2}\sqrt{4-x^2}$$

to calculate if we want to connect to the wifi. This is just the integral we did above - rearranging we get $x^2 + y^2 = 4$, a circle of radius 2. The bounds of the integral are -2 to 2 so it's a semicircle, giving us the wifi password as

$$\int_{-2}^{2}\left(x^3\cos\left(\frac{x}{2}\right)+\frac{1}{2}\right)\sqrt{4-x^2}\mathrm{d}x=\frac{1}{2}\cdot\frac{1}{2}\cdot 4\pi=\pi$$

If you didn't know any of this, π is a reasonable guess so you could do that too!

Exercises for this section:

1. Even functions are ones for which $f(x)=f(-x)$. In this case we have

$$\int_{-a}^{a} f(x)\mathrm{d}x = c\int_{0}^{a} f(x)\mathrm{d}x$$

for some constant c - what is c?

Reflections

Once we start thinking about areas, we can think about various geometric transformations and how they affect the integral and its value. Reflections down the middle of a region will preserve the area so the value of the integral will be the same. For an integral from a to b this corresponds to reflection in the line $x=\frac{a+b}{2}$. This will map a point t to $a+b-t$ so the function inside the integral will be changed from $f(x)$ to $f(a+b-x)$ which gives us the formula

$$\int_{a}^{b} f(x)\mathrm{d}x = \int_{a}^{b} f(a+b-x) \tag{4}$$

This is known as a reflection substitution and is a generalisation of the idea we had with odd functions - you could try prove that from this. And it's very powerful, this is a cornerstone of many integral calculations and often enough to solve problems by itself. The idea is that we take advantage of the symmetries of the function f to help us do the integral.

It's particularly useful on trigonometric functions which have a bunch of identities linking them to each other such as $\sin(\frac{\pi}{2}-x)=\cos(x)$, $\cos(\frac{\pi}{2}-x)=\sin(x)$ and $\tan(\frac{\pi}{2}-x)=\cot(x)$.

To demonstrate the use of this technique, consider this integral - Putnam 1980 A3:

$$I=\int_{0}^{\frac{\pi}{2}}\frac{\mathrm{d}x}{1+\tan^{\alpha}(x)} \tag{5}$$

where α is any real number. Since α can be anything, this integral looks really daunting. Using a reflection substitution,

$$I=\int_{0}^{\frac{\pi}{2}}\frac{\mathrm{d}u}{1+\frac{1}{\tan^{\alpha}}(u)}=\int_{0}^{\frac{\pi}{2}}\frac{\tan^{\alpha}(u)}{1+\tan^{\alpha}(u)}\mathrm{d}u$$

The integral has ended up in a very similar looking form - this is what usually happens with reflection substitutions - we get something of a similar form which we can then manipulate alongside our original integral. In this case, if we add the two integrals together, we get

$$2I=\int_{0}^{\frac{\pi}{2}}\mathrm{d}x=\frac{\pi}{2}$$

so that $I=\dfrac{\pi}{4}$. Not so bad for a Putnam problem! Putnam is known as one of the hardest math exams you can take - it's for undergraduates in the USA taken in two 3 hour parts. It has 12 questions in total with each being worth 10. The difficulty is so high that the median mark is generally 0 or 1.

Another integral which would be difficult to do otherwise is

$$I=\int_{-1}^{1}\frac{|x|}{1+e^{x}}\mathrm{d}x$$

Doing a reflection substitution, we get

$$I = \int_{-1}^{1} \frac{|x|}{1+e^{-x}}\mathrm{d}x = \int_{-1}^{1} \frac{e^x|x|}{1+e^x}\mathrm{d}x$$

and then

$$2I = \int_{-1}^{1} |x|\mathrm{d}x$$

after which it can be seen quite easily with a graph that $I = \frac{1}{2}$. This technique is really versatile and can help in simplifying a lot of problems.
Exercises for this section

1. Evaluate $\displaystyle\int_0^{2\pi} \sin(\sin(x) - x)\mathrm{d}x$
2. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin x^{\cos^{\sin x} x} - \cos x^{\sin^{\cos x} x}\mathrm{d}x$
3. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \log(\tan x)\mathrm{d}x$
4. (Putnam 1987 B1) Evaluate $\displaystyle\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$

Scaling

Another geometric transformation which affects areas is scaling by some factor s. A vertical stretch will just multiply every value by s, it's the well known fact that

$$\int_a^b sf(x)\mathrm{d}x = s\int_a^b f(x)\mathrm{d}x$$

The more interesting case is a horizontal stretch by a factor of s. The area will be increased by a factor of s again and the region of integration will be stretched to be from sa to sb. Then the point $(x, f(x))$ will be mapped to the point $(sx, f(x))$ so our function changes to $f\left(\frac{x}{s}\right)$. Therefore we have

$$\int_{sa}^{sb} f\left(\frac{x}{s}\right)\mathrm{d}x = s\int_a^b f(x)\mathrm{d}x \tag{6}$$

This can be used to evaluate the integral of $\frac{1}{x}$. Let

$$f(a) = \int_1^a \frac{\mathrm{d}x}{x}$$

Then we have that

$$f(ab) = \int_1^{ab} \frac{\mathrm{d}x}{x} \ \& \ f(b) = \int_1^b \frac{\mathrm{d}x}{x} = \int_a^{ab} \frac{\mathrm{d}x}{x}$$

using the above result. Therefore we see that

$$f(ab) = f(a) + f(b)$$

This was the argument used historically and it was recognised that this is a logarithm - previously they had only been calculation tools but now had theoretical importance too. $f(a)$ is in fact sometimes used as a definition for the logarithm in some treatments of analysis.

Finally, all these techniques can be combined to evaluate the log sine integral, my favourite integral.

$$I = \int_0^{\frac{\pi}{2}} \log(\sin x)\mathrm{d}x \tag{7}$$

With the reflection substitution we get

$$I = \int_0^{\frac{\pi}{2}} \log(\cos(x))\mathrm{d}x$$

A consequence of this is (one of the previous exercises!)

$$\int_0^{\frac{\pi}{2}} \log(\tan x)\mathrm{d}x = 0$$

Adding these two together, we get

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \log(\sin(x)\cos(x))\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \log(\sin(2x)) - \log(2)\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \log(\sin(2x))\mathrm{d}x - \frac{\pi \log 2}{2} \end{aligned}$$

In the integral, scale this horizontally by a factor of $\frac{1}{2}$. Then the integral becomes

$$\frac{1}{2}\int_0^{\pi} \log(\sin(u))\mathrm{d}u$$

Thinking about the values $\sin(x)$ takes on the interval $[\frac{\pi}{2}, \pi]$, it's the same as the interval $[0, \pi]$ so that

$$\int_0^{\frac{\pi}{2}} \log(\sin(u))\mathrm{d}u = \int_{\frac{\pi}{2}}^{\pi} \log(\sin(u))\mathrm{d}u$$

This means that the above becomes

$$2I = I - \frac{\pi \log 2}{2}$$

which gives us $I = -\dfrac{\pi \log 2}{2}$. This integral is a very common one and can be evaluated in many ways but none as simple as this.

Exercises for this section

1. Evaluate $\displaystyle\int_0^{\pi} x \ln(\sin x)\mathrm{d}x$
2. Given that

$$\int_0^{\infty} \frac{\ln x}{x^2+1}\mathrm{d}x = 0$$

   use a scaling argument to find the value of

$$\int_0^{\infty} \frac{\ln x}{x^2+a^2}\mathrm{d}x$$