Gamma Function talk

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1 Introduction

The Gamma Function is a special function which is the result of generalising the factorial to complex arguments. It's known as the 'least special of the special functions' because it commonly appears throughout mathematics but still isn't really something which undergraduate courses do much on (only in FCM and Asymptotic Methods here) which is a shame because it has really interesting properties and links together a lot of mathematics. Hopefully this talk will show that to you and contribute to making it less of a 'special function'!

2 The forms of the Gamma Function

We begin with the limit

$$\lim_{n\to\infty} \frac{n!(n+1)^m}{(n+m)!} = 1$$

for a fixed integer m. To justify it, expanding it out,

$$\begin{aligned}
\lim_{n\to\infty} \frac{n!(n+1)^m}{(n+m)!} &= \lim_{n\to\infty} \frac{(n+1)^m}{(n+1)\dots(n+m)} \\
&= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n+1}\right)\left(\frac{n+2}{n+1}\right)\dots\left(\frac{n+m}{n+1}\right)} \\
&= \left(\prod_{k=1}^{m} \lim_{n\to\infty} \frac{n+k}{n+1}\right)^{-1} = 1
\end{aligned}$$

because each of the limits in the product is 1 as m is a fixed integer.

To get an equation for $m!$ we can multiply by $m!$:

$$\lim_{n\to\infty} \frac{n!m!(n+1)^m}{(n+m)!} = m!$$

Manipulating it a bit,

$$\begin{aligned} m! &= \lim_{n\to\infty} \frac{m!(n+1)^z}{(n+m)!} \\ &= \lim_{n\to\infty} \frac{n!(n+1)^z}{(m+1)\dots(m+n)} \end{aligned}$$

This definition we can take to be our factorial for complex numbers - to emphasise this, replace m with $z \in \mathbb{C}$. Noting that both $\lim\limits_{n\to\infty} \frac{(n+1)^z}{n^z} = 1$ and that for historical reasons we shift down the argument, define the Gamma Function to be

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\dots(z+n)}$$

This is known as the Gauss Limit form of the Gamma Function.

From here we can derive other forms of the Gamma function. Replacing the n^z with $(n+1)^z$, noting that

$$(n+1)^z = \frac{2^z}{1^z} \cdot \frac{3^z}{2^z} \dots \frac{(n+1)^z}{n^z} = \prod_{k=1}^{n} \left(1 + \frac{1}{n}\right)^z$$

and dividing the top and bottom by $n!$ we get

$$\begin{aligned} \Gamma(z) &= \frac{1}{z} \frac{\prod_{k=1}^{n}(1+\frac{1}{n})^z}{(\frac{z+1}{1})(\frac{z+2}{2})\dots(\frac{z+n}{n})} \\ &= \frac{1}{z} \prod_{k=1}^{n} \frac{(1+\frac{1}{k})^z}{1+\frac{z}{k}} \end{aligned}$$

This is known as Euler's product form of the Gamma Function (we haven't proved yet that this is the same as the integral - it could be a different function with very similar properties - but let's assume that for now)

From the Gauss Limit we can derive another form of the Gamma Function. Start by writing $z^n = e^{z\log n}$ and divide by $n!$

$$\begin{aligned} \Gamma(z) &= \frac{1}{z} \lim_{n\to\infty} \frac{e^{z\log n}}{(\frac{z+1}{1})(\frac{z+2}{2})\dots(\frac{z+n}{n})} \\ &= \frac{1}{z} \lim_{n\to\infty} \frac{e^{z\log n}}{n} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1} \\ &= \lim_{n\to\infty} \frac{e^{z\log n - \sum_{k=1}^{n} \frac{1}{k}}}{z} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)^{-1} e^{\frac{z}{k}} \\ &= \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)^{-1} e^{\frac{z}{k}} \end{aligned}$$

This is the Weierstrass product representation of the Gamma function. The product representations are quite useful for expressions for $\log \Gamma(z)$ which can sometimes be easier to deal with than the Gamma function itself.

Now for a useful characterisation of the Gamma Function:

Theorem (Bohr-Mollerup):

If f satisfies the following properties for all complex x

1. $f(1) = 1$
2. $f(x+1) = xf(x)$
3. f is log convex

Then $f(x) = \Gamma(x)$

Proof: Since we have log convexity, it's reasonable to consider $g = \log f$. By the convexity of g, we have

$$\log n \leq \frac{g(x+n+1) - g(n+1)}{x} \leq \log(n+1) \tag{1}$$

for some integer n. An integer n was chosen because now we can use the functional equation given by point 2 repeatedly:

$$g(x+1) = \log x + g(x) \implies g(n+1+x) = g(x) + \log(x(x+1)\dots(x+n))$$

Putting this into (1) we get

$$0 \leq g(x) - \log\left(\frac{n!n^x}{x(x+1)\dots(x+n)}\right) \leq x \log\left(1 + \frac{1}{n}\right)$$

As $n \to \infty$ (take n to be increasingly large integers as it's true for all integers), g is determined uniquely and we get the result

$$f(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\dots(x+n)}$$

An integral you might remember from STEP is

$$f(n) = \int_0^\infty x^{n-1} e^{-x} \mathrm{d}x$$

Both $f(1) = 1$ and $f(n+1) = nf(n)$ can be verified fairly easily making it a candidate Gamma Function, the only thing left to prove is log convexity.

This can be done using Hölder's inequality, which states

$$\left|\int fg\mathrm{d}x\right| \leq \left(\int |f|^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int |f|^q \mathrm{d}x\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Using this on f,

$$\begin{aligned} f(ax + (1-a)y) &= \int_0^\infty t^{ax+(1-a)y} e^{-t} \mathrm{d}t \\ &= \int_0^\infty t^{ax+(1-a)y} e^{-ta} e^{-t(1-a)} \mathrm{d}t \\ &= \left(\int_0^\infty (e^{-ta} t^{ax})^{\frac{1}{a}} \right)^a \left(\int_0^\infty (e^{-t(1-a)} t^{(1-a)y})^{\frac{1}{1-a}} \right)^(1-a) \\ &= f(x)^a f(x)^{1-a} \end{aligned}$$

Taking the log of this,

$$\log f(ax + (1-a)y \leq a \log f(x) + (1-a) \log f(y)$$

which shows f is log convex so it's the Gamma function! This is the usual form you encounter the Gamma function in.

It makes sense that all the different forms of Γ ended up being the same - they all must be logarithmically convex. The first two are a strict requirement to be the same as the factorial function on the integers so only the third property really matters. Other functions satisfying the first two properties can be constructed such as $h(x) = e^{k \sin(2m\pi x)} \Gamma(x)$ for some integer m but these won't satisfy the log convexity. There are other theorems which characterise the Gamma Function as a unique function satisfying a property or the 'simplest' function satisfying certain properties, making it a natural choice [1]. It also has a lot of interesting properties which justifies studying it above other extensions too!

3 Stirling's Formula

The integral form of Γ can be used to give us Stirling's Formula with Laplace's Method. To set up the use of this, beginning with the integral,

$$\begin{aligned} M! &= \int_0^\infty e^{-x} x^M \mathrm{d}x (x = Nz) \\ &= \int_0^\infty e^{-Mz} (Mz)^M M \mathrm{d}z \\ &= M^{M+1} \int_0^\infty e^{-Mz} z^M \mathrm{d}z \\ &= M^{M+1} \int_0^\infty e^{M(\ln z - z)} \mathrm{d}z \end{aligned}$$

If M is large then the main contribution will be around the maximum of this integrand - other values have much smaller contributions. The maximum occurs

[1] https://mathoverflow.net/questions/23229/importance-of-log-convexity-of-the-gamma-function

at the maximum of $f(z) = \ln z - z$ which is at $z_0 = 1$, when the function takes the value $f(1) = -1$. Expanding in a Taylor series about this point,

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \dots \\ &= f(z_0) + \frac{f''(z_0)}{2}(z - z_0)^2 + \dots \end{aligned}$$

$f''(z) = -\frac{1}{z^2}$ so $f''(1) = -1$. Higher order terms in the Taylor series will contribute little because of M being large so we can approximate,

$$\begin{aligned} M! &\approx M^{M+1} \int_0^\infty e^{-M - M\frac{(z-1)^2}{2}} \mathrm{d}z \\ &= M^{M+1} e^{-M} \sqrt{2\pi M} = \sqrt{2\pi M} \left(\frac{M}{e}\right) \end{aligned}$$

This approximation is really accurate - 99% accuracy when n=10. It also shows us that the integral form of factorials is useful to deduce properties of factorials.

4 Beta Function

The main use of the Gamma Function when calculating integrals is through the Beta Function.

$$B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} \mathrm{d}x$$

where it's shifted down just like the Gamma Function. A useful expression for the Beta Function can be derived by multiplying two Gamma function integrals together:

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \left(\int_0^\infty e^{-t} t^{x-1} \mathrm{d}t\right) \left(\int_0^\infty e^{-s} s^{y-1} \mathrm{d}s\right) \\ &= \int_0^\infty \int_0^\infty e^{-t-s} t^{x-1} s^{y-1} \mathrm{d}t\mathrm{d}s \end{aligned}$$

Now in this double integral, use the change of variables $t = ru$ and $s = r(1-u)$. The Jacobean is r and this gives us

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^1 e^{-r}(ru)^{x-1}(r(1 - u))^{y-1} r\mathrm{d}r\mathrm{d}u$$

These can now be separated as a product of two integrals as the r integral doesn't depend on u at all:

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-r} r^{x+y-1} \int_0^1 u^{x-1}(1 - u)^{y-1} \mathrm{d}u = \Gamma(x + y)B(x, y)$$

Therefore

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

This relation is one of the main uses of either function.

Another immediate application of this relation is the Legendre Duplication Formula. Beginning with $B(z, \frac{1}{2})$, we have

$$\begin{aligned} B(z, \tfrac{1}{2}) &= \int_0^1 x^{-\frac{1}{2}}(1-x)^{z-1}\mathrm{d}x \\ &= 2\int_0^1 (1-u^2)^{z-1}\mathrm{d}u \\ &= 2\int_0^1 (1-u)^{z-1}(1+u)^{z-1}\mathrm{d}u \end{aligned}$$

Where in the first line the substitution $x = u^2$ was used. Using a reflection substitution, we get

$$\begin{aligned} B(z, \tfrac{1}{2}) &= 2\int_0^1 u^{z-1}(2-u)^{z-1}\mathrm{d}u \\ &= 2^z \int_0^1 u^{z-1}(1-\tfrac{u}{2})^{z-1}\mathrm{d}u \end{aligned}$$

This is starting to look like a Beta function integral but not quite - substitute $u = 2t$ which gives

$$B(z, \tfrac{1}{2}) = 2^{2z}\int_0^{\frac{1}{2}} t^{z-1}(1-t)^{z-1}\mathrm{d}t$$

Now thinking about the graph of this integrand f (or using the substitution $v = 1-t$) we get

$$\int_0^{\frac{1}{2}} f = \int_{\frac{1}{2}}^1 f$$

so we can extend the domain of the (equation number) to $[0, 1]$ which gives us

$$B(z, \tfrac{1}{2}) = 2^{2z-1}\int_0^1 t^{z-1}(1-t)^{z-1}\mathrm{d}t = 2^{2z-1}B(z,z)$$

Writing this in terms of Gamma functions and rearranging, we get Legendre's duplication formula:

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

This is a more precise version of the following result which comes up every now and then:

$$\frac{1}{2^{2n}}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}$$

The Bohr Mollerup Theorem can be used to derive pretty much all of the Gamma function identities as shown in Emil Artin's amazing book The Gamma Function. His proof of the reflection formula avoids the use of contour integration so is probably more interesting for you to see:

Consider the function

$$\varphi(x) = \Gamma(x)\Gamma(1-x)\sin \pi x$$

This has period 1 because

$$\begin{aligned}\varphi(x+1) &= \Gamma(x+1)\Gamma(-x)\sin(\pi(x+1)) \\ &= x\Gamma(x)\frac{\Gamma(1-x)}{-x}(-\sin \pi x) = \varphi(x)\end{aligned}$$

Next Legendre's duplication formula states

$$\Gamma(\tfrac{x}{2})\Gamma(\tfrac{x+1}{2}) = 2\sqrt{\pi}2^{-x}\Gamma(x)$$

and putting $1-x$ in it gives

$$\Gamma(\tfrac{1-x}{2})\Gamma(1-\tfrac{x}{2}) = 2\sqrt{\pi}2^{x-1}\Gamma(1-x)$$

Putting these to use,

$$\begin{aligned}\varphi(\tfrac{x}{2})\varphi(\tfrac{x+1}{2}) &= \Gamma(\tfrac{x}{2})\Gamma(1-\tfrac{x}{2})\sin(\tfrac{\pi x}{2})\Gamma(\tfrac{x+1}{2})\Gamma(\tfrac{1-x}{2})\cos\tfrac{\pi x}{2} \\ &= \pi\Gamma(x)\Gamma(1-x)\sin \pi x\end{aligned}$$

so we have

$$\varphi(\tfrac{x}{2})\varphi(\tfrac{x+1}{2}) = \pi\varphi(x) \tag{2}$$

Since $\Gamma(x)\&\sin(\pi x)$ are continuous and positive on $[0,1]$, φ is too. φ's periodicity means that these are both true for all $x \in \mathbb{R}$. This lets us define $gx = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\varphi(x)$. From equation 2, we get

$$\frac{1}{4}(g(\tfrac{x}{2}) + g(\tfrac{x+1}{2})) = g(x)$$

Now as g is continuous on $[0,1]$ it is bounded on that interval, periodicity lets us say that $g(x) \leq M$ for some $M \in \mathbb{R}$ for all $x \in \mathbb{R}$. Now we have

$$|g(x)| \leq \frac{1}{4}|g(\tfrac{x}{2})| + \frac{1}{4}|g(\tfrac{x+1}{2})| \leq \frac{M}{2}$$

So we can replace M by $\frac{M}{2}$. By infinite descent we can take $M = 0$ so that $g = 0$ so that $\log \varphi(x)$ is a linear function. However it's a linear periodic function so it's constant. How do we determine that constant? Well from the functional equation for Gamma, we get

$$\begin{aligned}\varphi(x) &= \frac{\Gamma(1+x)}{x}\Gamma(1-x)\sin \pi x \\ &= \Gamma(1+x)\Gamma(1-x)\left(\pi - \frac{\pi^3 x^2}{3!} + \dots\right)\end{aligned}$$

This shows that for continuity we should define $\phi(0) = \pi$ which is our constant and resolves ϕ being undefined at the integers! (otherwise we could have just put in $x = \frac{1}{2}$ which I wanted to do).

Now we can get an interesting product formula for sine:

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= \frac{e^{-\gamma z}}{z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1} e^{\frac{z}{k}} \cdot (-z)\frac{e^{\gamma z}}{-z}\prod_{k=1}^{\infty}\left(1-\frac{z}{k}\right)^{-1} e^{-\frac{z}{k}} \\ &= \frac{1}{z}\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)^{-1}\end{aligned}$$

From the reflection formula we then get

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right)$$

This is like if we pretended that we could factor sine like a polynomial, with its roots at each integer point. This is actually what Euler did when he claimed this result and it was later justified by Weierstrass with his Weierstrass factorisation theorem.

Though this result is interesting in itself, Euler came across it while trying to solve the problem which made him famous, the Basel problem, namely evaluating $\sum_{n=1}^{\infty} \frac{1}{n^2}$. He managed to come up with some integral expression of the series and approximate that to greater accuracy than his contemporaries. From there, he might have recognised the answer is $\frac{\pi^2}{6}$ which would give an advantage to solving the problem. The method he used was to claim this factorisation of sine. Then you can expand out the the coefficients of the product:

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) = 1 - x^2\sum_{k=1}^{\infty}\frac{1}{k^2} + \dots$$

Now this can be compared with the Taylor series of sine:

$$\frac{\sin(\pi x)}{\pi x} = 1 - \frac{\pi^2 x^2}{6} + \dots$$

Comparing the coefficients we get the result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

which is quite nice. Sadly this method doesn't generalise to the other values of $\zeta(2n)$ - that'll be tackled next!

5 Digamma Function

A useful alternative to studying the Gamma function is looking at its logarithm - similarly looking at the derivative of its logarithm can be better than looking at the derivative directly. This is known as the Digamma function $\psi(z)$

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}(\ln \Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}$$

A lot of the identities of the digamma function can be found directly from the ones from the Gamma function:

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{3}$$

$$\psi(1-x) - \psi(x) = \pi \cot \pi x \tag{4}$$

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty} z^n (-1)^n \zeta(n+1) \tag{5}$$

As usual, taking the logarithm of a product is a good idea and in this case, the product forms of the Gamma function give some useful identities.

In particular, starting with the Euler product representation,

$$\log \Gamma(z) = -\log z + \sum_{k=1}^{\infty} z \log\left(1 + \frac{1}{k}\right) - \log\left(1 + \frac{z}{k}\right) \implies \psi(z) = -\frac{1}{z} + \sum_{k=1}^{\infty} \log\left(1 + \frac{1}{k}\right) - \frac{1}{z+k}$$

This can be manipulated to give a series more like the Harmonic series by considering partial sums:

$$\sum_{k=1}^{n} \log\left(1 + \frac{1}{k}\right) = \ln(n+1) \approx -\gamma + \sum_{k=1}^{n} \frac{1}{k}$$

with this approximation becoming exact as $n \to \infty$. So we can rewrite the above series as

$$\psi(z) = -\frac{1}{z} - \gamma + \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{z+k}$$

Notably with $z = 1$ we get $\Gamma'(1) = -\gamma$.

As a side note, differentiating under the integral sign with this gives us

$$\int_0^\infty e^{-x} \ln x \mathrm{d}x = -\gamma$$

which is quite hard to get directly!

This series representation can be combined with the reflection formula for the digamma function to give us an interesting infinite series. First, note that by the functional equation,

$$\psi(1-x) = -\gamma + \sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k-x}$$

Then

$$\begin{aligned} \pi\cot(\pi x) = \psi(1-x) - \psi(x) &= \frac{1}{x} + \sum_{k=1}^{\infty} \frac{1}{k+x} - \frac{1}{k-x} \\ &= \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2x}{x^2-k^2} \implies \frac{1}{2x^2} + \sum_{k=1}^{\infty} \frac{1}{x^2-k^2} = \frac{\pi\cot(\pi x)}{2x} \end{aligned}$$

Extending the series from $-\infty$ to ∞, we can write this more neatly as

$$\sum_{k=-\infty}^{\infty} \frac{1}{x^2-k^2} = \frac{\pi\cot(\pi x)}{x}$$

This series can be derived using contour integration - viewed that way we can derive the reflection formula by working back with this method! Even more interesting is what happens when you pull this series apart as a geometric series:

$$\begin{aligned} \frac{1}{2x^2} + \sum_{k=1}^{\infty} \frac{1}{x^2-k^2} = \frac{1}{2x^2} - \sum_{k=1}^{\infty} \frac{\frac{1}{k^2}}{1-\frac{x^2}{k^2}} &= \frac{1}{2x^2} - \sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{x^2}{k^4} + \dots \\ &= \frac{1}{2x^2} - \sum_{k=1}^{\infty}\sum_{r=0}^{\infty} \frac{x^{2r}}{k^{2r+2}} \\ &= \frac{1}{2x^2} - \sum_{r=0}^{\infty} x^{2r}\zeta(2r+2) = \frac{\pi\cot(\pi x)}{2x} \end{aligned}$$

Now if we expand the terms of cot as a series, we can get values of $\zeta(2n)$ which is very nice! A good place to start from is the definition of Bernoulli numbers in terms of the generating function:

$$\frac{t}{e^t-1} = \frac{t}{2}\left(\coth\frac{t}{2} - 1\right) = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}$$

Then putting in $t = ix$, we get

$$\frac{ix}{2}\left(\coth\frac{ix}{2} - 1\right) = \frac{x}{2}\left(\cot\frac{x}{2} - i\right) = \sum_{n=0}^{\infty}\frac{B_n(ix)^n}{n!}$$

Splitting this into real and imaginary parts gives us the Taylor series for cot:

$$\cot x = \sum_{n=0}^{\infty}\frac{B_{2n}(-1)^n 2^{2m} x^{2n-1}}{(2n)!}$$

This is nice but why are the numbers B_n special enough to start from there? It won't be proved here but the Bernoulli numbers B_n appear in a few places such as Faulhaber's Formula:

$$d\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p + \sum_{k=2}^{p}\frac{B_k p!}{k!(p-k+1)!}n^{p-k+1}$$

and the Euler Maclaurin Summation formula

$$\sum_{i=m}^{n} f(i) = \int_m^n f(x)\mathrm{d}x + \frac{f(n)+f(m)}{2} + \sum_{k=1}^{\lfloor \frac{p}{2}\rfloor}\frac{B_{2k}}{(2k)!}(f^{(2k-1)}(n) - f^{(2k-1)}(m)) + R_p$$

for some remainder term R_p which can be estimated. This formula can be used for various things such as proving Faulhaber's formula and studying the asymptotic behaviour of functions like the digamma function.

Getting back to the zeta function, we have

$$\frac{\pi\cot(\pi x)}{2x} = \sum_{n=0}^{\infty}\frac{B_{2n}(-1)^n\pi^{2n}2^{2n-1}x^{2n-2}}{(2n)!} = \frac{1}{2x^2} - \sum_{n=1}^{\infty}x^{2n-2}\zeta(2n)$$

Since $B_0 = 1$, the first terms match up and from there, comparing coefficients, we get

$$\zeta(2n) = \frac{B_{2n}(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}$$

giving us every single one of them in one go. All the even integer values of the zeta function can be determined this way but not the odd values of the zeta function. The odd values of the zeta function in contrast are a complete mystery with not only no formula linking all of them but no known formula for any of them. Very little is known about most of the values with only $\zeta(3) \approx 1.20206$ being proved to be irrational by Roger Apéry in 1979.

6 Conclusion

I hope you enjoyed the talk - a quick tour through the most important properties of the Gamma Function and related objects and applications to various problems. This didn't cover a lot of areas it appears in such as in the calculation of integrals, the zeta function's functional equation (and so analytic number theory) and the Euler Mascheroni constant (also called gamma, γ!) so if you're interested those are good things to look into. In particular I recommend Emil Artin's book 'The Gamma Function' for a fairly rigorous approach to the Gamma Function while any decent book on integration will cover the calculating integrals aspect of the Gamma Function - Inside Interesting Integrals is great (my favourite math book!) or even the book I'm currently writing!

7 If there is time

7.1 The Gamma Zeta identity

The Gamma function can be nicely tied with the Zeta function in the following integral, due to Riemann in 1859:

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} \mathrm{d}x$$

To handle this one, the denominator is similar to the sum of a geometric series in form but we can't use it yet as if the ratio was e^x, the series would be divergent. So dividing by e^x on top and bottom to make the ratio e^{-x} and expressing it as a series gives:

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} \mathrm{d}x = \int_0^\infty \frac{x^{s-1}e^{-x}}{1 - e^{-x}} \mathrm{d}x = \int_0^\infty x^{s-1}e^{-x} \sum_{k=0}^\infty e^{-kx} = \sum_{k=1}^\infty \int_0^\infty x^{s-1}e^{-kx} \mathrm{d}x$$

The integral is very close to a Gamma Function in form now; subbing in $u = kx$, it becomes

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} \mathrm{d}x = \sum_{k=1}^\infty \int_0^\infty \left(\frac{u}{k}\right)^{s-1} e^{-u} \frac{\mathrm{d}x}{k} = \sum_{k=1}^\infty \frac{1}{k^s} \int_0^\infty u^{s-1}e^{-u} \mathrm{d}u = \Gamma(s)\zeta(s)$$

This identity is the part of deriving the Riemann Zeta functional equation which is how the zeta function is extended to the complex plane - the basis of the Riemann Hypothesis.

7.2 Gauss Multiplication Formula

The Bohr Mollerup theorem is really good at proving things are the Gamma function and sometimes also finding new identities for the Gamma function. We know that $\Gamma(x)$ is log convex so a bunch of products of them is also log convex. For example, we can consider a product of similar Gamma functions (inspired by the Legendre Duplication formula which appears later on):

$$f(x) = \Gamma\left(\frac{x}{p}\right)\Gamma\left(\frac{x+1}{p}\right)\ldots\Gamma\left(\frac{x+p-1}{p}\right)$$

Then we want $f(x+1) = f(x)$ and here we have

$$\begin{aligned} f(x+1) &= \Gamma\left(\frac{x+1}{p}\right)\dots\Gamma\left(\frac{x+p-1}{p}\right) \\ &= \Gamma\left(\frac{x}{p}\right)\dots\Gamma\left(\frac{x+p-1}{p}\right) \end{aligned}$$

which isn't quite the functional equation. We can modify it though to have the right property:

$$g(x) = p^x\Gamma\left(\frac{x}{p}\right)\Gamma\left(\frac{x+1}{p}\right)\dots\Gamma\left(\frac{x+p-1}{p}\right)$$

Now this is a Gamma function other than some normalisation factor by the Bohr Mollerup theorem. To calculate $g(1)$, use the Gauss Limit form of the Gamma Function, we have

To calculate $g(1)$ we can use the Euler reflection formula and the product

$$\prod_{r=1}^{n-1}\sin\left(\frac{r\pi}{n}\right) = \frac{n}{2^{n-1}}$$

There are two cases, when n is even and when n is odd, they're fairly similar overall but work out a little differently. First take n to be even, $p = 2k$. Then we get by pairing them up to use the reflection formula, (if this is put in the gamma function section, take out the derivation of the half sine product from the log sine section and put it here)

$$\begin{aligned} \Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}\right)\dots\Gamma\left(\frac{p-1}{p}\right) &= \Gamma\left(\frac{1}{2k}\right)\Gamma\left(\frac{2k-1}{2k}\right)\dots\Gamma\left(\frac{1}{2}\right) \\ &= \frac{\pi}{\sin\left(\frac{\pi}{2k}\right)}\dots\sqrt{\pi} \\ &= \pi^{k-\frac{1}{2}}\left(\prod_{r=1}^{k-1}\sin\left(\frac{r\pi}{2k}\right)\right)^{-1} \\ &= \pi^{k-\frac{1}{2}}\left(\frac{\sqrt{p}}{2^{k-\frac{1}{2}}}\right)^{-1} \\ &= \frac{(2\pi)^{k-\frac{1}{2}}}{\sqrt{2k}} \end{aligned}$$

For the odd case $p = 2k+1$ it's quite similar except the pairing doesn't give you the $\Gamma(\frac{1}{2})$ term, there is an exact pairing:

$$\begin{aligned}\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}\right)\ldots\Gamma\left(\frac{p-1}{p}\right) &= \Gamma\left(\frac{1}{2k+1}\right)\Gamma\left(\frac{2k}{2k+1}\right)\ldots\Gamma\left(\frac{k}{2k+1}\right)\Gamma\left(\frac{k+1}{2k+1}\right)\\ &= \pi^k \frac{\pi}{\sin\left(\frac{\pi}{2k+1}\right)}\cdots\frac{\pi}{\sin\left(\frac{k\pi}{2k+1}\right)}\\ &= \pi^k\left(\prod_{r=1}^{k}\sin\left(\frac{r\pi}{2k}\right)\right)^{-1}\\ &= \pi^k\left(\frac{\sqrt{p}}{2^k}\right)^{-1}\\ &= \frac{(2\pi)^k}{\sqrt{2k+1}}\end{aligned}$$

So in general, we have

$$\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{2}{p}\right)\ldots\Gamma\left(\frac{p-1}{p}\right) = \frac{(2\pi)^{\frac{p-1}{2}}}{\sqrt{p}}$$

Therefore by the Bohr Mollerup theorem we get

$$\Gamma\left(\frac{x}{p}\right)\ldots\Gamma\left(\frac{x+p-1}{p}\right) = (2\pi)^{\frac{p-1}{2}}p^{\frac{1}{2}-x}\Gamma(x)$$

This is known as Gauss' Multiplication formula. Note that with $p = 2$ we recover Legendre's duplication formula. Many of the other properties of the Gamma function can be derived using this theorem - often if you know what to look for - as can be seen in Artin's book The Gamma Function.