How To Make Convergent Series Out Of Divergent Series

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1 Introduction

Abel said 'Divergent series are in general something fatal, and it is a disgrace to base any proof on them' - this stance is reasonable. Prior to Abel, mathematicians like Euler had used divergent series to derive results, some of which were true and some of which weren't. These results could be proved with other techniques which weren't built upon shaky foundations so divergent series could be seen as a way to come up with plausible answers which can later be proven rigorously. A more interesting way to treat them however, is to generalise the idea of summation to give an answer to some divergent series. This turns out to be a useful approach which has appeared throughout mathematics and has even been observed in physics experiments!

2 Using a divergent series

We begin with mentioning the Basel Problem, namely evaluating the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1}$$

Euler had worked this out which brought him great fame. It's possible he did this by first working out a method to approximate the series so that it converges more quickly and then recognising the sum as $\frac{\pi^2}{6}$. This method is known as Euler Maclaurin summation, a fairly complicated looking formula:

$$\sum_{i=m}^{n} f(i) = \int_m^n f(x)\mathrm{d}x + \frac{f(n)+f(m)}{2} + \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(n) - f^{(2k-1)}(m)) + R_p$$

where B_k are the Bernoulli numbers, a sequence of rationals which appear when summing the n^{th} powers (Faulhaber's formula). This formula is essentially taking the integral approximation of a sum and adding in more and more terms with higher derivatives to make it more accurate - it can be proved by messing around.

After realising the answer, he tried to use trigonometric functions and link them in some way to this sum. This led to the well known proof using the factorisation of sine which he lacked justification for (this came much later with Weierstrass' product theorem in Complex Analysis over 100 years later). Euler may have cared about this as he tried to prove this result again and again - chances are he didn't and just wanted a method which generalised to other values of the zeta function - which he did manage with the even values. No one has managed to generalise this any further though!

One of the other methods of solving the Basel Problem which he came up with is through considering the series

$$\frac{1 - r\cos x}{1 - 2r\cos x + r^2} = 1 + r\cos x + r^2\cos 2x + r^3\cos 3x + \dots \tag{2}$$

which holds for $|r| < 1$. Sadly this method doesn't achieve a completely rigorous solution to the Basel Problem - it's in a section about divergent series! The series looks quite nasty but you might recognise that

$$1 - 2r\cos x + r^2 = (1 - re^{ix})(1 - re^{-ix})$$

by thinking about complex conjugates. Then this series is

$$\begin{aligned}\frac{1 - r\cos x}{1 - 2r\cos x + r^2} &= \Re\left(\frac{1 - re^{ix}}{(1 - re^{ix})(1 - re^{-ix})}\right) \\ &= \Re\left(\frac{1}{1 - re^{-ix}}\right) \\ &= \Re(1 + re^{-ix} + r^2e^{-2ix} + \dots) \\ &= 1 + r\cos x + r^2\cos 2x + \dots\end{aligned}$$

which is the answer we wanted. This is valid for $|r| < 1$ as we used a geometric series expansion but if we pretend we can put $r = 1$ into this series, we get

$$-\frac{1}{2} = \cos x + \cos 2x + \cos 3x + \dots \tag{3}$$

This is clearly divergent; if we put in $x = \pi$ then we get

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \dots \tag{4}$$

If you've seen the Numberphile video on $1 + 2 + 3 + \dots = -\frac{1}{12}$, this is the same result they got through something more ridiculous than this. This series is called Grandi's Series, keep it in mind!

The series (3) is quite useful; if we integrate it term by term from $x \in [0, \pi]$ to π

$$\frac{\pi - x}{2} = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots \tag{5}$$

This is actually true - it's a Fourier Series - and is valid for $x \in [0, \pi]$[1].

Now we can integrate from $[0, \pi]$, noting that

$$\int_0^\pi \frac{\pi - x}{2}\mathrm{d}x = \frac{\pi^2}{4}$$
$$\frac{1}{2n+1}\int_0^\pi \sin((2n+1)x)\mathrm{d}x = \frac{2}{(2n+1)^2}$$
$$\frac{1}{2n}\int_0^\pi \sin(2nx)\mathrm{d}x = 0$$

which gives us

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \tag{6}$$

which can be manipulated to give the Basel Problem. While it's great to get this series, it wasn't done in a way which was fair. However, we did get (4), a finite answer for a divergent series. How do we make sense of this?

3 Assigning finite values to divergent series

(4) claims that

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \dots$$

If we were Abel, we'd just throw this away without a second thought. But since we're not, let's see if we can make sense of this. A reasonable first step to examining this series is through its partial sums, they go

$$S_1 = 1, S_2 = 0, S_3 = 1, S_4 = 0, \dots$$

i.e they're 1 for odd terms in the sequence and 0 for even ones. So surely if we wanted to pick some number to say is the value of the series, we pick something between 0 and 1. What do we pick? Since 0 and 1 are reached equally, we should pick the number in the middle - there's no bias towards either. This is $\frac{1}{2}$ as we got before. However, this is also Numberphile reasoning. How do we make this into something reasonable?

[1] This can be seen by noticing the LHS isn't periodic but the RHS is. To change to other intervals of length π, just integrate on some interval $[x, (n+1)\pi]$ where $x \in [n\pi, (n+1)\pi]$

We could take the ideas from this and make it into a definition. Another way of going about the same thing is to take the average of the partial sums. This is known as Cesàro summation.

That is, for a sequence $(a_k)_{k=1}^{\infty}$, if we define the partial sums as

$$S_n = \sum_{k=1}^{n} a_k$$

Then we define the Cesàro sum to be the average of the partial sums:

$$S = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} S_k \tag{7}$$

If S exists, we say the series is Cesàro summable. From above, the Cesàro sum of Grandi's series is $\frac{1}{2}$ so we know that's Cesàro summable but not convergent. Does convergent imply Cesàro summability?

A reasonable argument for yes is that if a series is convergent, $S_k \to a$ for some number a. Then the average of the partial sums will get closer and closer to a because these terms dominate the average.

This can be worked into a proof, fixing $\varepsilon > 0$ if we take N large enough such that for $n \geq N$, $|S_n - a| < \varepsilon$ then we have

$$\begin{aligned}
\left| \frac{1}{n} \sum_{k=1}^{n} S_k - a \right| &= \left| \frac{\sum_{k=1}^{N-1} S_k - Na}{n} + \frac{\sum_{k=N}^{n} S_k - (n-N)a}{n} \right| \\
&\leq \left| \frac{\sum_{k=1}^{N-1} S_k - Na}{n} \right| + \frac{\sum_{k=N}^{n} |S_k - a|}{n} \\
&\leq \left| \frac{\sum_{k=1}^{N-1} S_k - Na}{n} \right| + \frac{(n-N)\varepsilon}{n}
\end{aligned}$$

The first term is a constant over n so we can choose n large enough so that term is less than ε and then we're done.

So we have that convergence implies Cesàro summability and that they converge to the same value. So Cesàro summation is a strictly stronger sense of summation than normal summation. This gives some justifcation for equation (4).

However we can't do every series; if the average partial sum is too big then the Cesàro sum won't converge. An example is the Numberphile series;

$$1 + 2 + 3 + \cdots = -\frac{1}{12} \tag{8}$$

Here, $S_n = \dfrac{n(n+1)}{2} = O(n^2)$ which is too big. You can probably guess how to give meaning to this, another method of summation.

The method of summation in question is Ramanujan Summation, a fairly scary looking formula:

$$f(1)+f(2)+f(3)+\cdots=-\frac{f(0)}{2}+i\int_0^\infty\frac{f(it)-f(-it)}{e^{2\pi t}-1}\mathrm{d}t \tag{9}$$

He came up with this starting with the Euler Maclaurin summation formula, it also bears a lot of resemblance to another result known as the Abel-Plana formula:

$$\sum_{n=0}^{\infty}f(n)=\frac{1}{2}f(0)=\int_0^\infty f(x)\mathrm{d}x+i\int_0^\infty\frac{f(it)-f(-it)}{e^{2\pi t}-1}\mathrm{d}t \tag{10}$$

which can be proved using contour integration. We can recover Grandi's series with this; if we choose a function $f(x)=(-1)^x$ for integer x and 0 otherwise, we get the same result as (4) immediately. We could extend this function to $\mathbb{C}$ and get a different answer - a problem with Ramanujan summation. It can be defined in an alternative way if the derivation is modified so that the resulting formula gives one answer only but historically, this is the formula Ramanujan used. He didn't actually distinguish in his notebook that it was a new type of summation, it was written literally! It does agree with the sum when it's convergent though.

Another series whose answer can be seen quite easily is $1+1+1+\dots$, take $f(n)=1$ and then the integral cancels out, leaving

$$1+1+1+\cdots=-\frac{1}{2} \tag{11}$$

This summation method does give an answer to the series $1+2+3+\dots$ Take $f(n)=n$ and we get

$$1+2+3+\cdots=2\int_0^\infty\frac{t}{1-e^{2\pi t}}\mathrm{d}t$$

It's reasonable to expect this integral to exist as there's an exponential term in the denominator. To evaluate it, knowing the Gamma function, having exponentials and powers in an integral is doable. Therefore we should expand the bottom as a geometric series to get a sum of integrals we know:

$$\begin{aligned}
2\int_0^\infty \frac{t}{1-e^{2\pi t}}\mathrm{d}t &= -2\int_0^\infty \frac{te^{-2\pi t}}{1-e^{-2\pi t}}\mathrm{d}t \\
&= -\frac{1}{2\pi^2}\int_0^\infty \frac{ue^{-u}}{1-e^{-u}}\mathrm{d}u (u = 2\pi t) \\
&= -\frac{1}{2\pi^2}\int_0^\infty \sum_{n=0}^\infty ue^{-(n+1)u}\mathrm{d}u \\
&= -\frac{1}{2\pi^2}\sum_{n=0}^\infty \frac{1}{(n+1)^2}\int_0^\infty ve^{-v}\mathrm{d}v (v = (n+1)u) \\
&= -\frac{1}{2\pi^2}\left(\sum_{n=1}^\infty \frac{1}{n^2}\right)\left(\int_0^\infty ve^{-v}\mathrm{d}v\right) \\
&= -\frac{1}{12}
\end{aligned}$$

where in the last line, the Basel Problem we solved came back to help us! For the integral, you can recognise it as $\Gamma(2) = 1! = 1$ or just do by parts. This is the answer we were hoping for and this value actually has special meaning beyond these summation methods which will be explored in the next section.

4 Appearances of sums of divergent series

These methods turn out to not just be mathematical curiosities but point to deeper theory. Ramanujan communicated his evaluation of (8) to Hardy in his original letter, to which Hardy thought it was remarkable that Ramanujan had discovered special cases of the value of the zeta function without any knowledge of Complex Analysis. That is, defining the zeta function as

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \tag{12}$$

it can be analytically continued with some contour integration by deriving the functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{13}$$

Using this, we extend the definition of the zeta function to the entire complex plane; (12) is only defined for $\Re(s) > 1$. Then we get values such as $\zeta(0) = -\frac{1}{2}$ and $\zeta(-1) = -\frac{1}{12}$. We can't equate (12) with the results from the functional equation but if you want to assign a finite value to the series when $Re(s) \leq 1$ then it's reasonable to take

$$\sum_{n=1}^\infty \frac{1}{n^s} = \zeta(s)$$

This is in agreement with the results from Ramanujan summation (8) and (11) which is why Hardy was very impressed and why Numberphile manipulated the results until they got the numbers they wanted.

These values also appear in the context of Physics, a lot of Quantum Physics needs to use a technique called Renormalization to assign finite values to results which are divergent. An example of this is the Casimir effect which is about the force between two plates because of the quantum fluctuations of the field. (include a picture)

In the calculation of this effect, the following evaluation is made

$$\sum_{n=1}^{\infty} n^3 = \zeta(-3) = \frac{1}{120}$$

There are many similar examples where techniques to treat infinities are required but despite the unsatisfactory nature mathematically, such results have been confirmed by experiment. The Casimir effect was first proposed in 1948 and wasn't verified until 1997 to within 5% of the value predicted by the theory.

5 Appendices

Here are some other sections I wrote but doubt I'd have time to talk about.

5.1 Abel Summation

There is another type of summation named after Abel ironically - maybe his studies with divergent series is what made him think they're the 'invention of the devil'. This method is based on using power series expansions; we define

$$\sum_{n=0}^{\infty} a_n = \lim_{z \to 1^-} \sum_{n=0}^{\infty} a_n z^n \tag{14}$$

and take this to be our sum. This lets us calculate Grandi's series too, using the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

Putting $z = 1$ in, we recover the answer we expect. Abel summation is in fact stronger than Cesàro summation - if something is Cesàro summable then it is Abel summable with the same answer.

However, there are series which are Abel summable but not Cesàro summable. For example, the series

$$1 - 2 + 3 - 4 + 5 - 6 + \dots \tag{15}$$

Here the partial sums are

$$S_{2k+1} = 2k, S_{2k} = -1$$

so $\sum_{k=1}^{n} S_k = O(n^2)$ so it isn't Cesàro summable. However, to calculate the Abel sum, start from the series

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$$

Differentiating, we get

$$\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - 4z^3 + \dots$$

Therefore the Abel sum is

$$1 - 2 + 3 - 4 + \dots = \frac{1}{4} \tag{16}$$

5.2 Calculating the log sine integral

My favourite integral is

$$\int_0^{\frac{\pi}{2}} \log(\sin x)\mathrm{d}x = -\frac{\pi}{2}\log 2 \tag{17}$$

It's not for any particular reason other than it being a fairly nice integral which can be done in many clever ways - so far I'm on 12 haha. It can also be done with a divergent series.

Begin with a series similar to (2), take the geometric series and put $z = e^{ix}$ into it. Then

$$1 + e^{ix} + e^{2ix} + \dots = \frac{1}{1 - e^{ix}} = \frac{1}{2} + \frac{i}{2}\cot(x)$$

Taking the imaginary part, we get

$$1 + \sin x + \sin 2x + \dots = \frac{1}{2}\cot\left(\frac{x}{2}\right) \tag{18}$$

This is a divergent series; the terms don't go towards 0 in general. Ignoring that and multiplying by x and integrating from 0 to $\frac{\pi}{2}$, after observing that

$$\int_0^{\frac{\pi}{2}} x\sin(2nx)\mathrm{d}x = \frac{\pi(-1)^n}{4n}$$

we get

$$\int_0^{\frac{\pi}{2}} x\cot x\mathrm{d}x = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \frac{\pi}{2}\log 2$$

Now we can integrate by parts to get

$$\int_0^{\frac{\pi}{2}} x \cot x \mathrm{d}x = [x \log(\sin x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log(\sin x) \mathrm{d}x$$

Therefore we conclude that

$$\int_0^{\frac{\pi}{2}} \log(\sin x) \mathrm{d}x = -\frac{\pi}{2} \log 2 \tag{19}$$

5.3 Eilenburg-Mazur swindle

While these infinite sums aren't valid in the context of summing numbers, in other contexts, they can make sense and constitute a valid proof. This is known as the Eilenburg-Mazur swindle. It was originally used in Geometric Topology and Algebra in ways I can't understand but another use of it is to prove the Cantor-Schroder-Bernstein theorem which states that if there is an injection of sets between X and Y and Y and X then there is a bijection between x and Y.

We can write this with a sum notation; an injectino between X and Y means there are two sets A and B such that

$$X = Y + A, Y = X + B$$

where the + means a disjoint union and the = means there is a bijection between the two sets. Then substituting these into each other, we have

$$X = X + A + B \tag{20}$$

Now we let the set Z be the elements on the left hand side of (18) which correspond to an element of the right. Then we can expand this indefinitely to give

$$X = A + B + A + B + \cdots + Z \tag{21}$$

Substitution $Y = B + X$, we get

$$Y = B + A + B + A + \cdots + Z$$

Since there is a bijection $A + B = B + A$ we can switch every $B + A$ in the above and get

$$Y = A + B + A + B + \cdots + Z \tag{22}$$

Now we can compose the bijection for X and Y in (19) and (20) to get the result that $X = Y$. So 'divergent' series can be useful in other areas too!

6 Further Reading

If you were interested in this, there's a lot of places to go for further reading. A very good book despite its age is Hardy's Divergent Series where he explores a lot more of the theory behind this, particularly Abelian and Tauberian theorems (theorems which tell you about whether the sum gives the same answer as the convergent series and if it's summable with a given method then it's summable in the usual sense, respectively). Another method of summation to look into is Borel Summation which is particularly useful for summing divergent asymptotic series. I hope you enjoyed the talk!

7 Bibliography

1. Hardy, G.H. (1949) Divergent series. Oxford: Clarendon.
2. Delabaere, 2011, Ramanujan's Summation, Available here
3. Brilliant.org, Sums Of Divergent Series, Available here